the bodies examined by them, and obtaining only negative results, and always a certain result with a crystal of the salt, they have insisted that this is the only nucleus. Others, again, have sought for an explanation in some catalytic or other mysterious force; while a third set of observers have declared it to be a matter of uncertainty or hazard whether a foreign body acts as a nucleus or not. In reviewing the subject and repeating my experiments in various ways, I see no reason for withdrawing from the theory which I had the honour of submitting to the Society eleven years ago, namely, that the action of nuclei is simply mechanical, and is capable of being expressed by the familiar word adhesion.

VIII. "On Definite Integrals involving Elliptic Functions." By J. W. L. GLAISHER, M.A., F.R.S., Fellow of Trinity College, Cambridge. Received July 31, 1879.

§ 1. The chief object of this paper is to apply to definite integrals involving elliptic functions certain special methods which have been employed for the evaluation of integrals of a similar kind involving circular functions.

§ 2. One of the most elegant and direct investigations of the value of the integral

$$\int_0^{\frac{1}{2}\pi} \log \sin x \, dx = \tfrac{1}{2}\pi \log (\tfrac{1}{2})$$

is afforded by the product

$$\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{(n-1)\pi}{2n} = \sqrt{n} \,.\, 2^{-\frac{1}{2}(n-1)},$$

for, taking the logarithm of both sides of this equation, and writing $\frac{\pi}{n}=h$,

$$h(\log\sin h + \log\sin 2h \ldots + \log\sin\tfrac{1}{2}\pi) = \pi \lim._{n=\infty} \frac{\log\{\sqrt{n} \,.\, 2^{-\frac{1}{2}(n-1)}\}}{n}$$
$$= -\tfrac{1}{2}\pi \log 2.$$

The same principle also gives the value of the integral

$$\int_0^{\pi} \log(1-2a\cos x + a^2)dx \quad \ldots\ldots \quad (1),$$

which $=0$ or $2\pi\log a$ according as $a <$ or > 1, and it is easy to see in general that if

$$\phi\left(a+\frac{b}{n}\right)\phi\left(a+\frac{2b}{n}\right)\ldots\phi\left(a+\frac{pnb}{n}\right)=\psi(n),$$

then $$\int_a^{a+pb}\log\phi(x)dx = b\lim._{n=\infty}\frac{\log\psi(n)}{n} = b\frac{\psi'(\infty)}{\psi(\infty)}.$$

For example, from the product

$$\Gamma(a)\Gamma\left(a+\frac{1}{n}\right)\ldots\Gamma\left(a+\frac{n-1}{n}\right)=\Gamma(na)\,.\,(2\pi)^{\frac{1}{2}(n-1)}n^{\frac{1}{2}-na},$$

since, when n is great,

$$\Gamma(na)=(2\pi)^{\frac{1}{2}}(na)^{na-\frac{1}{2}}e^{-na},$$

we derive the evaluation,

$$\int_a^{a+1}\log\Gamma(x)\,dx=\tfrac{1}{2}\log(2\pi)+a(\log a-1).$$

The number of products that yield integrals of interest is not great, and those just noticed are all that I remember to have seen applied to this purpose. The transformation formulæ in elliptic functions lead, however, in this manner, immediately to definite integrals, as will appear in the next section.

§ 3. Writing sn, cn, dn in place of sin am, cos am, Δ am, we have ("Fundamenta Nova," p. 46):—

$$\{\operatorname{sn}2\omega\operatorname{sn}4\omega\ldots\operatorname{sn}(n-1)\omega\}^2=\frac{(-1)^{\frac{1}{2}(n-1)}}{M}\left(\frac{\lambda}{k^n}\right)^{\frac{1}{2}}\quad.\quad.\quad(2).$$

$$\{\operatorname{cn}2\omega\operatorname{cn}4\omega\ldots\operatorname{cn}(n-1)\omega\}^2=\left(\frac{\lambda k'^n}{\lambda' k^n}\right)^{\frac{1}{2}}\quad.\quad.\quad.\quad.\quad.\quad.\quad(3).$$

$$\{\operatorname{dn}2\omega\operatorname{dn}4\omega\ldots\operatorname{dn}(n-1)\omega\}^2=\left(\frac{k'^n}{\lambda'}\right)^{\frac{1}{2}}.\quad.\quad.\quad.\quad.\quad.\quad.\quad(4).$$

In the case of the first real transformation (n being an uneven prime)

$$\omega=\frac{K}{n},\qquad \Lambda=\frac{K}{nM},\qquad \Lambda'=\frac{K'}{M}.$$

Thus, when n is infinite,

$$\Lambda=\tfrac{1}{2}\pi,\qquad M=\frac{2K}{n\pi},\qquad \Lambda'=\tfrac{1}{2}n\frac{\pi K'}{K},$$

but $\Lambda'=\log\frac{4}{\lambda}$, so that $\qquad \lambda=e^{-\frac{1}{2}n\frac{\pi K'}{K}}$,

and we therefore have

$$\log\left(\frac{\lambda}{k^n}\right)^{\frac{1}{2}}=-\tfrac{1}{4}n\frac{\pi K'}{K}-\tfrac{1}{2}n\log k,$$

$$\log\left(\frac{\lambda k'^n}{\lambda' k^n}\right)^{\frac{1}{2}}=-\tfrac{1}{4}n\frac{\pi K'}{K}+\tfrac{1}{2}n\log\left(\frac{k'}{k}\right)-\tfrac{1}{2}\log(\tfrac{1}{2}\pi),$$

$$\log\left(\frac{k'^n}{\lambda'}\right)^{\frac{1}{2}}=\tfrac{1}{2}n\log k'-\tfrac{1}{2}\log(\tfrac{1}{2}\pi),$$

whence the products (2), (3), (4), give

$$\int_0^{\mathrm{K}} \log \operatorname{sn} x\,dx = -\tfrac{1}{4}\pi\mathrm{K}' - \tfrac{1}{2}\mathrm{K}\log k \quad . \quad . \quad . \quad . \quad (5),$$

$$\int_0^{\mathrm{K}} \log \operatorname{cn} x\,dx = -\tfrac{1}{4}\pi\mathrm{K}' + \tfrac{1}{2}\mathrm{K}\log\left(\frac{k'}{k}\right) \quad . \quad . \quad . \quad . \quad (6),$$

$$\int_0^{\mathrm{K}} \log \operatorname{dn} x\,dx = \tfrac{1}{2}\mathrm{K}\log k' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

which are the analogues in elliptic functions of the integrals

$$\int_0^{\frac{1}{2}\pi} \log \sin x\,dx = \tfrac{1}{2}\pi\log\left(\tfrac{1}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

$$\int_0^{\frac{1}{2}\pi} \log \cos x\,dx = \tfrac{1}{2}\pi\log\left(\tfrac{1}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

The other formulæ, such as, *ex. gr.*,

$$\operatorname{sn} u \operatorname{sn}\{u+4\omega\} \ldots \operatorname{sn}\{u+4(n-1)\omega\} = \left(\frac{\lambda}{k^n}\right)^{\frac{1}{2}} \operatorname{sn}\left(\frac{u}{\mathrm{M}}, \lambda\right),$$

do not lead to new integrals, for this product gives

$$\int_0^{4\mathrm{K}} \log \operatorname{sn}(u+x)\,dx = -\pi\mathrm{K}' - 2\mathrm{K}\log k,$$

in which the sign of sn $(u+x)$ when negative is to be changed, so that the quantity under the integral sign should be written $\frac{1}{2}\log \operatorname{sn}^2(u+x)$. This result, however, may be readily deduced from (5).

The remaining formulæ on page 51 of the "Fundamenta Nova" only produce the equation

$$\int_0^{\mathrm{K}} \tfrac{1}{2}\log\left(1-\frac{\operatorname{sn}^2 u}{\operatorname{sn}^2 x}\right)^2 dx = \int_0^{\mathrm{K}} \log\left(1-k^2\operatorname{sn}^2 u \operatorname{sn}^2 x\right)dx,$$

which may be otherwise deduced from the formulæ (44) and (45) of § 10.

§ 4. In the second real transformation

$$\omega = \frac{i\mathrm{K}'}{n}, \qquad \lambda_1 = \tfrac{1}{2}\pi, \qquad \lambda_1' = e^{-\frac{1}{2}n\frac{\pi\mathrm{K}}{\mathrm{K}'}},$$

so that we find

$$\int_0^{\mathrm{K}'} \tfrac{1}{4}\log \operatorname{sn}^4 ix\,dx = -\tfrac{1}{2}\mathrm{K}'\log k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

$$\int_0^{\mathrm{K}'} \log \operatorname{cn} ix\,dx = \tfrac{1}{4}\pi\mathrm{K} + \tfrac{1}{2}\mathrm{K}'\log\left(\frac{k'}{k}\right) \quad . \quad . \quad . \quad . \quad (11),$$

$$\int_0^{\mathrm{K}'} \log \operatorname{dn} ix\,dx = \tfrac{1}{4}\pi\mathrm{K} + \tfrac{1}{2}\mathrm{K}'\log k' \quad . \quad . \quad . \quad . \quad . \quad (12),$$

but if, in virtue of the formulæ

$$\operatorname{sn} ix = i\,\frac{\operatorname{sn}(x, k')}{\operatorname{cn}(x, k')},$$

$$\operatorname{cn} ix = \frac{1}{\operatorname{cn}(x, k')},$$

$$\operatorname{dn} ix = \frac{\operatorname{dn}(x, k')}{\operatorname{cn}(x, k')},$$

these equations be written in a real form, they only reproduce (5), (6), and (7), as might have been anticipated *à priori.*

§ 5. The double products ("Fundamenta Nova," p. 66)

$$\Pi \operatorname{sn}^2 \frac{2m\mathrm{K}+2m'i\mathrm{K}'}{n} = \frac{(-)^{\frac{1}{2}(n-1)}n}{k^{\frac{1}{2}(n^2-1)}},$$

$$\Pi \operatorname{cn}^2 \frac{2m\mathrm{K}+2m'i\mathrm{K}'}{n} = \left(\frac{k'}{k}\right)^{\frac{1}{2}(n^2-1)},$$

$$\Pi \operatorname{dn}^2 \frac{2m\mathrm{K}+2m'i\mathrm{K}'}{n} = k'^{\frac{1}{2}(n^2-1)},$$

give rise to the remarkable integrals

$$\int_0^{\mathrm{K}}\int_{-\mathrm{K}'}^{\mathrm{K}'} \log \operatorname{sn}(x+iy)\,dxdy = -\mathrm{KK}'\log k \quad . \quad . \quad . \quad (13),$$

$$\int_0^{\mathrm{K}}\int_{-\mathrm{K}'}^{\mathrm{K}'} \log \operatorname{cn}(x+iy)\,dxdy = \mathrm{KK}'\log\left(\frac{k'}{k}\right) \quad . \quad . \quad . \quad (14),$$

$$\int_0^{\mathrm{K}}\int_{-\mathrm{K}'}^{\mathrm{K}'} \log \operatorname{dn}(x+iy)\,dxdy = \mathrm{KK}'\log k' \quad . \quad . \quad . \quad . \quad (15).$$

These may of course also be written

$$\int_0^{\mathrm{K}}\int_0^{\mathrm{K}'} \log\{\operatorname{sn}(x+iy)\operatorname{sn}(x-iy)\}\,dxdy = -\mathrm{KK}'\log k;\ \&c.$$

§ 6. Using the values of the three integrals in (5), (6), (7), and observing that

$$\int_0^{\mathrm{K}} \log \operatorname{sn} 2x\,dx = \tfrac{1}{2}\int_0^{2\mathrm{K}} \log \operatorname{sn} x\,dx,$$

$$= \tfrac{1}{2}\int_0^{\mathrm{K}} \log \operatorname{sn} x\,dx + \tfrac{1}{2}\int_{\mathrm{K}}^{2\mathrm{K}} \log \operatorname{sn} x\,dx,$$

$$= \tfrac{1}{2}\int_0^{\mathrm{K}} \log \operatorname{sn} x\,dx + \tfrac{1}{2}\int_0^{\mathrm{K}} \log \operatorname{sn}(2\mathrm{K}-x)\,dx,$$

$$= \int_0^{\mathrm{K}} \log \operatorname{sn} x\,dx,$$

we find, by means of the formula

$$\operatorname{sn} 2x = \frac{2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x}{1 - k^2 \operatorname{sn}^4 x},$$

that
$$\int_0^{\mathrm{K}} \log (1 - k^2 \operatorname{sn}^4 x) dx = -\tfrac{1}{4}\pi \mathrm{K}' + \mathrm{K} \log \left(\frac{2k'}{k^{\frac{1}{2}}}\right) \quad . \quad . \quad (16).$$

Similarly

$$\int_0^{\mathrm{K}} \tfrac{1}{2} \log \operatorname{cn}^2 2x dx = \int_0^{\mathrm{K}} \log \operatorname{cn} x dx,$$

$$\int_0^{\mathrm{K}} \log \operatorname{dn} 2x dx = \int_0^{\mathrm{K}} \log \operatorname{dn} x dx,$$

whence, from the formulæ

$$\operatorname{cn} 2x = \frac{1 - 2 \operatorname{sn}^2 x + k^2 \operatorname{sn}^4 x}{1 - k^2 \operatorname{sn}^4 x},$$

$$\operatorname{dn} 2x = \frac{1 - 2k^2 \operatorname{sn}^2 x + k^2 \operatorname{sn}^4 x}{1 - k^2 \operatorname{sn}^4 x},$$

by the use of (16), we find

$$\int_0^{\mathrm{K}} \tfrac{1}{2} \log (1 - 2 \operatorname{sn}^2 x + k^2 \operatorname{sn}^4 x)^2 dx = -\tfrac{1}{2}\pi \mathrm{K}' + \mathrm{K} \log \left(\frac{2k'^{\frac{3}{2}}}{k}\right) \quad . \quad . \quad (17).$$

$$\int_0^{\mathrm{K}} \log (1 - 2k^2 \operatorname{sn}^2 x + k^2 \operatorname{sn}^4 x) dx = -\tfrac{1}{4}\pi \mathrm{K}' + \mathrm{K} \log \left(\frac{2k'^{\frac{3}{2}}}{k^{\frac{1}{2}}}\right) \quad . \quad . \quad (18).$$

These two integrals may also be put respectively in the forms

$$\int_0^{\mathrm{K}} \tfrac{1}{2} \log (\operatorname{cn}^2 x - \operatorname{sn}^2 x \operatorname{dn}^2 x)^2 dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),$$

$$\int_0^{\mathrm{K}} \log (\operatorname{dn}^2 x - k^2 \operatorname{sn}^2 x \operatorname{cn}^2 x) dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20),$$

the former expression is written $\frac{1}{2} \log (\;\;)^2$, as the quantity in brackets is negative from $x = \frac{1}{2}\mathrm{K}$ to $x = \mathrm{K}$.

§ 7. Using the formulæ

$$\begin{aligned}
1 - \operatorname{cn} 2x &= 2\operatorname{sn}^2 x \operatorname{dn}^2 x && \div (1 - k^2 \operatorname{sn}^4 x), \\
1 + \operatorname{cn} 2x &= 2\operatorname{cn}^2 x && \div (1 - k^2 \operatorname{sn}^4 x), \\
1 - \operatorname{dn} 2x &= 2k^2 \operatorname{sn}^2 x \operatorname{cn}^2 x && \div (1 - k^2 \operatorname{sn}^4 x), \\
1 + \operatorname{dn} 2x &= 2\operatorname{dn}^2 x && \div (1 - k^2 \operatorname{sn}^4 x), \\
\operatorname{cn}^2 x &= (\operatorname{dn} 2x + \operatorname{cn} 2x) && \div (1 + \operatorname{dn} 2x), \\
\operatorname{dn}^2 x &= (k'^2 + \operatorname{dn} 2x + k^2 \operatorname{cn} 2x) && \div (1 + \operatorname{dn} 2x),
\end{aligned}$$

and, observing that

$$\int_0^{\mathrm{K}} \phi(2x) dx = \tfrac{1}{2} \int_0^{2\mathrm{K}} \phi(x) dx,$$

we have

$$\int_0^{2\mathrm{K}} \log(1 \pm \operatorname{cn} x)\,dx = -\tfrac{1}{2}\pi\mathrm{K}' - \mathrm{K}\log k \quad . \quad . \quad . \quad . \quad . \quad . \quad (21),$$

$$\int_0^{2\mathrm{K}} \log(1 - \operatorname{dn} x)\,dx = -\tfrac{3}{2}\pi\mathrm{K}' + \mathrm{K}\log k \quad . \quad . \quad . \quad . \quad . \quad (22),$$

$$\int_0^{2\mathrm{K}} \log(1 + \operatorname{dn} x)\,dx = \tfrac{1}{2}\pi\mathrm{K}' + \mathrm{K}\log k \quad . \quad . \quad . \quad . \quad . \quad (23),$$

$$\int_0^{2\mathrm{K}} \log(\operatorname{dn} x + \operatorname{cn} x)\,dx = -\tfrac{1}{2}\pi\mathrm{K}' + \mathrm{K}\log\left(\frac{k'^2}{k}\right) \quad . \quad . \quad . \quad (24),$$

$$\int_0^{2\mathrm{K}} \log(k'^2 + \operatorname{dn} x + k^2 \operatorname{cn} x)\,dx = \tfrac{1}{2}\pi\mathrm{K}' + \mathrm{K}\log(kk'^2) \quad . \quad (25).$$

Since $\operatorname{sn}(2\mathrm{K}-u) = \operatorname{sn} u$, $\operatorname{dn}(2\mathrm{K}-u) = \operatorname{dn} u$, $\operatorname{cn}(2\mathrm{K}-u) = -\operatorname{cn} u$, it follows that

$$\left.\begin{aligned} \int_0^{2\mathrm{K}} \phi(\operatorname{sn} x)\,dx &= 2\int_0^{\mathrm{K}} \phi(\operatorname{sn} x)\,dx \quad . \quad . \quad . \quad . \quad . \\ \int_0^{2\mathrm{K}} \phi(\operatorname{dn} x)\,dx &= 2\int_0^{\mathrm{K}} \phi(\operatorname{dn} x)\,dx \quad . \quad . \quad . \quad . \quad . \\ \int_0^{2\mathrm{K}} \phi(\operatorname{cn} x)\,dx &= \int_0^{\mathrm{K}} \{\phi(\operatorname{cn} x) + \phi(-\operatorname{cn} x)\}\,dx \end{aligned}\right\} \quad . \quad (26),$$

and therefore from (22) and (23) we deduce that

$$\int_0^{\mathrm{K}} \log(1 - \operatorname{dn} x)\,dx = -\tfrac{3}{4}\pi\mathrm{K}' + \tfrac{1}{2}\mathrm{K}\log k \quad . \quad . \quad . \quad . \quad (27),$$

$$\int_0^{\mathrm{K}} \log(1 + \operatorname{dn} x)\,dx = \tfrac{1}{4}\pi\mathrm{K}' + \tfrac{1}{2}\mathrm{K}\log k \quad . \quad . \quad . \quad . \quad (28).$$

But (21) only gives the value of

$$\int_0^{\mathrm{K}} \{\log(1 - \operatorname{cn} x) + \log(1 + \operatorname{cn} x)\}\,dx,$$

and is, therefore, equivalent to (5), while (24) and (25) are merely transformations of (5) and (28).

We also see that in (24) and (25) the integrals may be written

$$\int_0^{2\mathrm{K}} \log(\operatorname{dn} x - \operatorname{cn} x)\,dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29),$$

$$\int_0^{2\mathrm{K}} \log(k'^2 + \operatorname{dn} x - k^2 \operatorname{cn} x)\,dx \quad . \quad . \quad . \quad . \quad . \quad (30),$$

Applying (26) to (17) and (18) in the forms (19) and (20), we see that

$$\int_0^{2\mathrm{K}} \tfrac{1}{2}\log(\operatorname{cn} x \pm \operatorname{sn} x \operatorname{dn} x)^2\,dx = -\tfrac{1}{2}\pi\mathrm{K}' + \mathrm{K}\log\left(\frac{2k'^{\frac{3}{2}}}{k}\right). \quad . \quad . \quad (31),$$

$$\int_0^{2\mathrm{K}} \log(\operatorname{dn} x \pm k \operatorname{sn} x \operatorname{cn} x)\,dx = -\tfrac{1}{4}\pi\mathrm{K}' + \mathrm{K}\log\left(\frac{2k'^{\frac{3}{2}}}{k^{\frac{1}{2}}}\right). \quad . \quad . \quad (32).$$

§ 8. Since

$$\int_0^{\mathrm{K}} \phi(x)dx = \int_0^{\mathrm{K}} \phi(\mathrm{K}-x)dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33),$$

it follows from (27) and (28) that

$$\int_0^{\mathrm{K}} \log\,(\mathrm{dn}\,x - k')dx = -\tfrac{3}{4}\pi\mathrm{K}' + \tfrac{1}{2}\mathrm{K}\log\,(kk') \quad . \quad . \quad . \quad (34),$$

$$\int_0^{\mathrm{K}} \log\,(\mathrm{dn}\,x + k')dx = \tfrac{1}{4}\pi\mathrm{K}' + \tfrac{1}{2}\mathrm{K}\log\,(kk') \quad . \quad . \quad . \quad (35).$$

It may be remarked that the transformation (33) applied to the functions sn, cn, dn does not suffice to give the values of the integrals in (5) and (6), although we thus immediately obtain (7); for

$$\int_0^{\mathrm{K}} \log \mathrm{dn}\,xdx = \int_0^{\mathrm{K}} \log \mathrm{dn}\,(\mathrm{K}-x)dx = \int_0^{\mathrm{K}} \log k'dx - \int_0^{\mathrm{K}} \log \mathrm{dn}\,xdx,$$

giving

$$\int_0^{\mathrm{K}} \log \mathrm{dn}\,xdx = \tfrac{1}{2}\mathrm{K}\log k';$$

but

$$\int_0^{\mathrm{K}} \log \mathrm{sn}\,xdx = \int_0^{\mathrm{K}} \log \mathrm{cn}\,xdx - \int_0^{\mathrm{K}} \log \mathrm{dn}\,xdx,$$

and

$$\int_0^{\mathrm{K}} \log \mathrm{cn}\,xdx = \int_0^{\mathrm{K}} \log k'dx + \int_0^{\mathrm{K}} \log \mathrm{sn}\,xdx - \int_0^{\mathrm{K}} \log \mathrm{dn}\,xdx,$$

only lead to one equation between the integrals of $\log \mathrm{sn}\,x$ and $\log \mathrm{cn}\,x$, viz.,

$$\int_0^{\mathrm{K}} \log \mathrm{cn}\,xdx - \int_0^{\mathrm{K}} \log \mathrm{sn}\,xdx = \tfrac{1}{2}\mathrm{K}\log k'.$$

§ 9. Putting $x=y$ in the formula

$$\Theta(x+y)\Theta(x-y) = \frac{\Theta^2 x \Theta^2 y}{\Theta^2 0}(1-k^2\,\mathrm{sn}^2\,x\,\mathrm{sn}^2\,y) \quad . \quad . \quad . \quad (36),$$

it becomes

$$\Theta(2x) = \frac{\Theta^4 x}{\Theta^3 0}(1-k^2\,\mathrm{sn}^4\,x) \quad . \quad . \quad . \quad . \quad . \quad (37),$$

whence

$$\int_0^{\mathrm{K}} \log \Theta(2x)dx = 4\int_0^{\mathrm{K}} \log \Theta xdx - 3\int_0^{\mathrm{K}} \log \Theta 0dx + \int_0^{\mathrm{K}} \log\,(1-k^2\,\mathrm{sn}^4\,x)dx.$$

Now

$$\int_0^{\mathrm{K}} \log \Theta(2x)dx = \int_0^{\mathrm{K}} \log \Theta xdx,$$

so that the equation is

$$\int_0^{\mathrm{K}} \log \Theta xdx = \mathrm{K}\log \Theta 0 - \tfrac{1}{3}\int_0^{\mathrm{K}} \log\,(1-k^2\,\mathrm{sn}^4\,x)dx,$$

and substituting the value of the last-written integral from (16) and putting for $\Theta 0$ its value, viz. $\left(\frac{2k'\mathrm{K}}{\pi}\right)^{\frac{1}{2}}$, this gives

$$\int_0^{\mathrm{K}} \log \Theta x dx = \tfrac{1}{12}\pi\mathrm{K}' + \tfrac{1}{6}\mathrm{K} \log\left(\frac{2kk'\mathrm{K}^3}{\pi^3}\right) \quad . \quad . \quad . \quad . \quad (38).$$

Also $$\mathrm{H}x = \sqrt{k} \,.\, \mathrm{sn}\, x\Theta x,$$

and integrating the logarithm of this equation, using (5) and (38), we have

$$\int_0^{\mathrm{K}} \log \mathrm{H}x dx = -\tfrac{1}{6}\pi\mathrm{K}' + \tfrac{1}{6}\mathrm{K} \log\left(\frac{2kk'\mathrm{K}^3}{\pi^3}\right).$$

Replace x, y in (36) by mx, nx and the formula becomes

$$\Theta(m+n)x\Theta(m-n)x = \frac{\Theta^2(mx)\Theta^2(nx)}{\Theta^2 0}(1-k^2\,\mathrm{sn}^2\, mx\,\mathrm{sn}^2\, nx),$$

whence, since (p being any integer)

$$\int_0^{\mathrm{K}} \log \Theta(px)dx = \int_0^{\mathrm{K}} \log \Theta x dx,$$

we have

$$\int_0^{\mathrm{K}} \log(1-k^2\,\mathrm{sn}^2\, mx\,\mathrm{sn}^2\, nx)dx = 2\left(\mathrm{K}\log\Theta 0 - \int_0^{\mathrm{K}} \log\Theta x dx\right) \quad . \quad (39),$$

if m and n be different integers, and

$$= 3\left(\mathrm{K}\log\Theta 0 - \int_0^{\mathrm{K}} \log\Theta x dx\right) \quad . \quad (40),$$

if m and n be equal integers.

Thus we find

$$\int_0^{\mathrm{K}} \log(1-k^2\,\mathrm{sn}^2\, mx\,\mathrm{sn}^2\, nx)dx = -\tfrac{1}{6}\pi\mathrm{K}' + \tfrac{2}{3}\mathrm{K}\log\left(\frac{2k'}{k^{\frac{1}{2}}}\right) \quad . \quad (41).$$

if m and n be different integers, and

$$= -\tfrac{1}{4}\pi\mathrm{K}' + \mathrm{K}\log\left(\frac{2k'}{k^{\frac{1}{2}}}\right) \quad . \quad (42),$$

if m and n be equal integers.

The relation

$$\int_0^{\mathrm{K}} \log(1-k^2\,\mathrm{sn}^2\, mx\,\mathrm{sn}^2\, nx)dx = \tfrac{2}{3}\int_0^{\mathrm{K}} \log(1-k^2\,\mathrm{sn}^4\, mx)dx \quad . \quad . \quad (43)$$

is curious. The value of the latter integral is of course immediately derivable from (16).

§ 10. (*Lemma.*) If $\phi(x)$ be an even function of x, and if $\phi(x+2\mathrm{K}) = \phi(x)$, then

$$\int_0^{\mathrm{K}} \phi(x+y)dx + \int_0^{\mathrm{K}} \phi(x-y)dx = 2\int_0^{\mathrm{K}} \phi(x)dx,$$

for, if u denote the left hand member of this equation,

$$\begin{aligned}\frac{du}{dy}&=\int_0^{\mathrm{K}}\frac{d}{dy}\phi(x+y)dx+\int_0^{\mathrm{K}}\frac{d}{dy}\phi(x-y)dx,\\ &=\int_0^{\mathrm{K}}\frac{d}{dx}\phi(x+y)dx-\int_0^{\mathrm{K}}\frac{d}{dx}\phi(x-y)dx,\\ &=\phi(\mathrm{K}+y)-\phi(y)-\phi(\mathrm{K}-y)+\phi(-y),\\ &=0,\end{aligned}$$

so that u is independent of y and the lemma is proved. Now from (36),

$$\int_0^{\mathrm{K}}\log\Theta(x+y)dx+\int_0^{\mathrm{K}}\log\Theta(x-y)dx=2\int_0^{\mathrm{K}}\log\Theta x\,dx+2\mathrm{K}\log\Theta y$$
$$-2\mathrm{K}\log\Theta 0+\int_0^{\mathrm{K}}\log\left(1-k^2\,\mathrm{sn}^2\,x\,\mathrm{sn}^2\,y\right)dx,$$

and since Θx satisfies the conditions supposed in the lemma, the left-hand member of this equation is equal to the first term on the right-hand side, so that we obtain the formula

$$\int_0^{\mathrm{K}}\log\left(1-k^2\,\mathrm{sn}^2\,x\,\mathrm{sn}^2\,y\right)dx=2\mathrm{K}\log\frac{\Theta 0}{\Theta y}=\mathrm{K}\log\left(\frac{2k'\mathrm{K}}{\pi}\right)-2\mathrm{K}\log\Theta y \quad (44),$$

which we may write in the form

$$\int_0^{\mathrm{K}}\log\left(1-a^2\,\mathrm{sn}^2\,x\right)dx=\mathrm{K}\log\left(\frac{2k'\mathrm{K}}{\pi}\right)-2\mathrm{K}\log\Theta\left(\mathrm{sn}^{-1}\frac{a}{k}\right).$$

In virtue of the formula

$$\mathrm{sn}\,(x+y)\,\mathrm{sn}\,(x-y)=\frac{\mathrm{sn}^2\,x-\mathrm{sn}^2\,y}{1-k^2\,\mathrm{sn}^2\,x\,\mathrm{sn}^2\,y},$$

$$\mathrm{cn}\,(x+y)\,\mathrm{cn}\,(x-y)=\frac{1-\mathrm{sn}^2\,x-\mathrm{sn}^2\,y+k^2\,\mathrm{sn}^2\,x\,\mathrm{sn}^2\,y}{1-k^2\,\mathrm{sn}^2\,x\,\mathrm{sn}^2\,y},$$

$$\mathrm{dn}\,(x+y)\,\mathrm{dn}\,(x-y)=\frac{1-k^2\,\mathrm{sn}^2\,x-k^2\,\mathrm{sn}^2\,y+k^2\,\mathrm{sn}^2\,x\,\mathrm{sn}^2\,y}{1-k^2\,\mathrm{sn}^2\,x\,\mathrm{sn}^2\,y},$$

we can deduce by means of (44), since $\mathrm{sn}^2\,x$, $\mathrm{cn}^2\,x$, and $\mathrm{dn}\,x$ each satisfy the conditions of the lemma, that

$$\int_0^{\mathrm{K}}\tfrac{1}{2}\log(\mathrm{sn}^2\,x-\mathrm{sn}^2 y)^2dx=-\tfrac{1}{2}\pi\mathrm{K}'+\mathrm{K}\log\left(\frac{2k'\mathrm{K}}{k\pi}\right)-2\mathrm{K}\log\Theta y \quad . \quad (45),$$

$$\int_0^{\mathrm{K}}\tfrac{1}{2}\log\left(1-\mathrm{sn}^2\,x-\mathrm{sn}^2\,y+k^2\,\mathrm{sn}^2\,x\,\mathrm{sn}^2\,y\right)^2dx$$
$$=-\tfrac{1}{2}\pi\mathrm{K}'+\mathrm{K}\log\left(\frac{2k'^2\mathrm{K}}{k\pi}\right)-2\mathrm{K}\log\Theta y \quad . \quad (46),$$

$$\int_0^{\mathrm{K}} \log\,(1-k^2\,\mathrm{sn}^2\,x-k^2\,\mathrm{sn}^2\,y+k^2\,\mathrm{sn}^2\,x\,\mathrm{sn}^2\,y)\,dx$$
$$=\mathrm{K}\log\left(\frac{2k'^2\mathrm{K}}{\pi}\right)-2\mathrm{K}\log\Theta y \quad . \quad (47).$$

The last two integrals may also be written

$$\int_0^{\mathrm{K}} \tfrac{1}{2}\log\,(\mathrm{cn}^2\,x-\mathrm{dn}^2\,x\,\mathrm{sn}^2\,y)^2 dx,$$

$$\int_0^{\mathrm{K}} \log\,(\mathrm{dn}^2\,x-k^2\,\mathrm{cn}^2\,x\,\mathrm{sn}^2\,y)\,dx,$$

and if these be transformed by the substitution of $\mathrm{K}-x$ for x, they become respectively

$$\int_0^{\mathrm{K}} \tfrac{1}{2}\log\,(\mathrm{sn}^2\,x-\mathrm{sn}^2\,y)^2 dx-2\int_0^{\mathrm{K}} \log\,\mathrm{dn}\,x dx+2\mathrm{K}\log k',$$

$$\int_0^{\mathrm{K}} \log\,(1-k^2\,\mathrm{sn}^2\,x\,\mathrm{sn}^2\,y)\,dx-2\int_0^{\mathrm{K}} \log\,\mathrm{dn}\,x dx+2\mathrm{K}\log k',$$

so that of the four integrals (44), (45), (46), (47), the pair (44), (47) are convertible one with another, and also the pair (45), (46), by the substitution of $\mathrm{K}-x$ for x.

Integrating (44), . . . (47) with regard to y between the limits K and 0, and using (38) we obtain the following evaluations:—

$$\int_0^{\mathrm{K}}\!\!\int_0^{\mathrm{K}} \log\,(1-k^2\,\mathrm{sn}^2\,x\,\mathrm{sn}^2\,y)\,dxdy=-\tfrac{1}{6}\pi\mathrm{KK}'+\tfrac{2}{3}\mathrm{K}^2\log\left(\frac{2k'}{k^{\frac{1}{2}}}\right) \quad . \quad . \quad (48),$$

$$\int_0^{\mathrm{K}}\!\!\int_0^{\mathrm{K}} \tfrac{1}{2}\log\,(\mathrm{sn}^2\,x-\mathrm{sn}^2\,y)^2 dxdy=-\tfrac{2}{3}\pi\mathrm{KK}'+\tfrac{2}{3}\mathrm{K}^2\log\left(\frac{2k'}{k^2}\right) \quad . \quad . \quad (49),$$

$$\int_0^{\mathrm{K}}\!\!\int_0^{\mathrm{K}} \tfrac{1}{2}\log(1-\mathrm{sn}^2x-\mathrm{sn}^2y+k^2\,\mathrm{sn}^2x\,\mathrm{sn}^2y)^2 dxdy=-\tfrac{2}{3}\pi\mathrm{KK}'+\tfrac{2}{3}\mathrm{K}^2\log\left(\frac{2k'^{\frac{5}{2}}}{k^2}\right)$$
$$. \quad . \quad (50),$$

$$\int_0^{\mathrm{K}}\!\!\int_0^{\mathrm{K}} \log\,(1-k^2\,\mathrm{sn}^2\,x-k^2\,\mathrm{sn}^2\,y+k^2\,\mathrm{sn}^2\,x\,\mathrm{sn}^2\,y)\,dxdy$$
$$=-\tfrac{1}{6}\pi\mathrm{KK}'+\tfrac{2}{3}\mathrm{K}^2\log\left(\frac{2k'^{\frac{5}{2}}}{k^{\frac{1}{2}}}\right) \quad . \quad . \quad (51),$$

§ 11. The lemma at the beginning of the last section may be deduced directly from the definition of integration when y is real; for, in virtue of the equation $\phi(x+2\mathrm{K})=\phi(x)$, $\phi(x+y)+\phi(x-y)$ can always be reduced to the form $\phi(x+a)+\phi(x-a)$ where $a<\mathrm{K}$; and

$$\int_0^{\mathrm{K}} \phi(x+a)\,dx+\int_0^{\mathrm{K}} \phi(x-a)\,dx$$
$$=h\{\phi(a)+\phi(a+h)+\phi(a+2h)\ \ldots\ +\phi(a+\mathrm{K})$$
$$+\phi(-a)+\phi(-a+h)+\phi(-h)+\phi(0)+\phi(h)\ \ldots\ +\phi(\mathrm{K}-a)\}.$$

Since $\phi(\mathrm{K}+a-h)=\phi(2\mathrm{K}-\mathrm{K}-a+h)=\phi(\mathrm{K}+a+h)$, &c., and $\phi(-h)=\phi(h)$, &c., this

$$=h\{\phi(0)+\phi(h)\ \ldots\ +\phi(\mathrm{K}-a)+\phi(\mathrm{K}-a+h)\ \ldots\ +\phi(\mathrm{K})$$
$$+\phi(0)+\phi(h)\ \ldots\ +\phi(a)+\phi(a+h)\ \ldots\ +\phi(\mathrm{K})\}$$
$$=2\int_0^{\mathrm{K}}\phi(x)dx.$$

The lemma is thus evidently true for all real values of y, and from the proof in § 11 it would appear that in general it was probably true when y was imaginary or of the form $a+bi$. It is, however, certainly not true in the case of $y=2i\mathrm{K}'$, for

$$\Theta(x+2i\mathrm{K}')=-e^{\frac{\pi}{\mathrm{K}}(\mathrm{K}'-ix)}\Theta x.$$

so that $$\log\Theta(x+2i\mathrm{K}')+\log\Theta(x-2i\mathrm{K}')=\frac{2\pi\mathrm{K}'}{\mathrm{K}}+2\log\Theta x,$$

and therefore

$$\int_0^{\mathrm{K}}\log\Theta(x+2i\mathrm{K}')dx+\int_0^{\mathrm{K}}\log\Theta(x-2i\mathrm{K}')dx=2\pi\mathrm{K}'+2\int_0^{\mathrm{K}}\log\Theta x dx.$$

It is also evident that (44), . . . (47) cannot be universally true, for the left-hand members of these equations remain unaltered when y is increased by $2i\mathrm{K}'$, which is not the case with the right-hand members, since Θ is not periodic with respect to $2i\mathrm{K}'$.

To determine the imaginary values of y for which the lemma and the theorems (44), . . . (47) deduced from it are true, consider the expansion of $\log\Theta(x)$ in a series of cosines, viz.:

$$\log\Theta x=\mathrm{A}-\frac{2q}{1-q^2}\cos\frac{\pi x}{\mathrm{K}}-\tfrac{1}{2}\frac{2q^2}{1-q^4}\cos\frac{2\pi x}{\mathrm{K}}-\tfrac{1}{3}\frac{2q^3}{1-q^6}\cos\frac{3\pi x}{\mathrm{K}}-\ \&\text{c.}\quad(52),$$

where $$\mathrm{A}=\tfrac{1}{12}\frac{\pi\mathrm{K}'}{\mathrm{K}}+\tfrac{1}{6}\log\left(\frac{2kk'\mathrm{K}^3}{\pi^3}\right).$$

From this we obtain (38), viz.:

$$\int_0^{\mathrm{K}}\log\Theta x dx=\mathrm{AK},$$

since $$\int_0^{\mathrm{K}}\cos\frac{n\pi x}{\mathrm{K}}dx=0.$$

Now $\log\Theta(x+a)+\log\Theta(x-a)$

$$=2\mathrm{A}-\Sigma\frac{1}{n}\frac{2q^n}{1-q^{2n}}\left\{\cos\frac{n\pi(x+a)}{\mathrm{K}}+\cos\frac{n\pi(x-a)}{\mathrm{K}}\right\}$$

$$=2\mathrm{A}-\Sigma\frac{1}{n}\frac{4q^n}{1-q^{2n}}\cos\frac{n\pi x}{\mathrm{K}}\cos\frac{n\pi a}{\mathrm{K}},$$

so that the lemma will necessarily hold good so long as the series

$$\Sigma\frac{1}{n}\frac{4q^n}{1-q^{2n}}\cos\frac{n\pi x}{\mathrm{K}}\cos\frac{n\pi a}{\mathrm{K}}$$

is convergent.

Let $a=mi\mathrm{K}'$, then

$$\cos\frac{n\pi a}{\mathrm{K}}=\tfrac{1}{2}(q^{mn}+q^{-mn})=\tfrac{1}{2}\frac{1+q^{2mn}}{q^{mn}},$$

and the series

$$=\Sigma\frac{2}{n}\frac{1+q^{2mn}}{1-q^{2n}}q^{n(1-m)}\cos\frac{n\pi x}{\mathrm{K}},$$

which is convergent when $m<1$.

It follows therefore that the lemma and the theorems (44), . . . (47) are true when y is of the form $a+bi$, where b lies between K' and $-\mathrm{K}'$, and a is unrestricted.

In the case of $b=\mathrm{K}'$ the series becomes neutral, but it is easy to see that the lemma and theorems are still true; for, transform (44) by putting $y=i\mathrm{K}'+z$, so that

$$k\,\mathrm{sn}\,y=\frac{1}{\mathrm{sn}\,z},$$

then $\log\Theta y=\log\Theta(i\mathrm{K}'+z)$

$$=\log\{ik^{\frac{1}{2}}e^{\frac{\pi}{4\mathrm{K}}(\mathrm{K}'-2iz)}\,\mathrm{sn}\,z\Theta z\}$$

$$=\tfrac{1}{2}\log k+\tfrac{1}{4}\frac{\pi\mathrm{K}'}{\mathrm{K}}+\log\mathrm{sn}\,z+\log\Theta z+\log i-\frac{i\pi z}{2\mathrm{K}}.$$

Thus we have

$$\int_0^{\mathrm{K}}\log(\mathrm{sn}^2z-\mathrm{sn}^2x)\,dx=-\tfrac{1}{2}\pi\mathrm{K}'+\mathrm{K}\log\left(\frac{2k'\mathrm{K}}{k\pi}\right)-2\mathrm{K}\log\Theta z+i\pi(z-\mathrm{K}).$$

The imaginary term $i\pi(z-\mathrm{K})$ is due to the fact that $\mathrm{sn}^2z-\mathrm{sn}^2x$ changes sign when $x=z$, so that we might expect the term $(\mathrm{K}-z)\log(-1)$ to appear in the equation. Writing, therefore, the integral in the real form

$$\int_0^{\mathrm{K}}\tfrac{1}{2}\log(\mathrm{sn}^2z-\mathrm{sn}^2x)^2,$$

and throwing away the imaginary term, we have (45). It may be observed that (46) and (47) are also connected by the same substitution of $y=i\mathrm{K}'+z$.

§ 12. In the formula (44), viz.:

$$\int_0^{\mathrm{K}}\log(1-k^2\,\mathrm{sn}^2x\,\mathrm{sn}^2y)\,dx=2\mathrm{K}\{\log\Theta 0-\log\Theta y\},$$

put y equal to $\tfrac{1}{2}\mathrm{K}$, $\tfrac{1}{2}\mathrm{K}+i\mathrm{K}'$, $\tfrac{1}{2}i\mathrm{K}'$, $\mathrm{K}+\tfrac{1}{2}i\mathrm{K}'$, $\tfrac{1}{2}\mathrm{K}-\tfrac{1}{2}i\mathrm{K}'$, $\tfrac{1}{2}\mathrm{K}+\tfrac{1}{2}i\mathrm{K}'$ respectively; we thus find, on substituting for $\mathrm{sn}^2 y$ its value in the different cases

$$\int_0^{\mathrm{K}} \log\{1-(1-k')\,\mathrm{sn}^2 x\}dx \;=2\mathrm{K}\{\log\Theta 0-\log\Theta(\tfrac{1}{2}\mathrm{K})\},$$

$$\int_0^{\mathrm{K}} \log\{1-(1+k')\,\mathrm{sn}^2 x\}dx \;=2\mathrm{K}\{\log\Theta 0-\log\Theta(\tfrac{1}{2}\mathrm{K}+i\mathrm{K}')\},$$

$$\int_0^{\mathrm{K}} \log\{1+k\,\mathrm{sn}^2 x\}dx \;=2\mathrm{K}\{\log\Theta 0-\log\Theta(\tfrac{1}{2}i\mathrm{K}')\},$$

$$\int_0^{\mathrm{K}} \log\{1-k\,\mathrm{sn}^2 x\}dx \;=2\mathrm{K}\{\log\Theta 0-\log\Theta(\mathrm{K}+\tfrac{1}{2}i\mathrm{K}')\},$$

$$\int_0^{\mathrm{K}} \log\{1-k(k-ik')\,\mathrm{sn}^2 x\}dx =2\mathrm{K}\{\log\Theta 0-\log\Theta(\tfrac{1}{2}\mathrm{K}-\tfrac{1}{2}i\mathrm{K}')\},$$

$$\int_0^{\mathrm{K}} \log\{1-k(k+ik')\,\mathrm{sn}^2 x\}dx=2\mathrm{K}\{\log\Theta 0-\log\Theta(\tfrac{1}{2}\mathrm{K}+\tfrac{1}{2}i\mathrm{K}')\}.$$

To obtain the values of the thetas, put $x=\frac{1}{2}\mathrm{K}$, $y=\frac{1}{2}\mathrm{K}$ in (36), and this equation gives

$$\Theta^4(\tfrac{1}{2}\mathrm{K})=\frac{1+k'}{2k'}\Theta^3 0\,\Theta\mathrm{K},$$

whence

$$\Theta(\tfrac{1}{2}\mathrm{K})=\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k'^{\frac{1}{8}}(1+k')^{\frac{1}{4}} \quad . \; . \; . \; . \; . \; . \quad (53).$$

From this, by taking $x=\frac{1}{2}\mathrm{K}$ in the formula

$$\Theta(x+i\mathrm{K}')=ik^{\frac{1}{2}}q^{-\frac{1}{4}}e^{\frac{-i\pi x^2}{2\mathrm{K}}}\,\mathrm{sn}\,x\,\Theta x,$$

we deduce that

$$\Theta(\tfrac{1}{2}\mathrm{K}+i\mathrm{K}')=e^{\frac{\pi\mathrm{K}'}{4\mathrm{K}}}\,\frac{(1+i)\mathrm{K}^{\frac{1}{2}}}{2^{\frac{1}{4}}\pi^{\frac{1}{2}}}k'^{\frac{1}{8}}(1-k')^{\frac{1}{4}} \quad . \; . \; . \quad (54).$$

The value of $\Theta(\frac{1}{2}i\mathrm{K}')$ may be deduced from (53) by putting $x=\frac{1}{2}\mathrm{K}$ in the formula

$$\Theta x=\left(\frac{k'\mathrm{K}}{k\mathrm{K}'}\right)^{\frac{1}{2}}e^{-\frac{\pi x^2}{4\mathrm{KK}'}}\,\frac{1}{\mathrm{cn}\,x}\Theta(ix,k'),$$

and changing the modulus from k to k': it is thus found that

$$\Theta(\tfrac{1}{2}i\mathrm{K}')=e^{\frac{\pi\mathrm{K}'}{16\mathrm{K}}}\,\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}(1-k)^{\frac{1}{4}} \quad . \; . \; . \; . \; . \quad (55),$$

and thence that

$$\Theta(\mathrm{K}+\tfrac{1}{2}i\mathrm{K}')=e^{\frac{\pi\mathrm{K}'}{16\mathrm{K}}}\,\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}(1+k)^{\frac{1}{4}}.$$

Finally, from (36) by putting $x=\frac{1}{2}\mathrm{K}+\frac{1}{2}i\mathrm{K}'$, $y=\frac{1}{2}\mathrm{K}+\frac{1}{2}i\mathrm{K}'$, we find that

$$\Theta^4(\tfrac{1}{2}\mathrm{K}+\tfrac{1}{2}i\mathrm{K}')=\frac{\Theta^3 0\Theta(\mathrm{K}+i\mathrm{K}')}{2k'(k'-ik)}$$

$$=e^{\frac{\pi\mathrm{K}'}{4\mathrm{K}}}\frac{2\mathrm{K}^2}{\pi^2}k^{\frac{1}{2}}k'^{\frac{1}{2}}(k'+ik)\quad .\ .\ .\ .\ .\ .\quad (56).$$

Substituting these values in the integrals and reducing, we have

$$\int_0^{\mathrm{K}}\log\{1-(1-k')\,\mathrm{sn}^2 x\}\,dx=\tfrac{1}{2}\mathrm{K}\log\left\{\frac{2k'^{\frac{3}{2}}}{1+k'}\right\}\quad .\ .\ .\ .\ .\ .\ .\quad (57),$$

$$\int_0^{\mathrm{K}}\tfrac{1}{2}\log\{1-(1+k')\,\mathrm{sn}^2 x\}^2 dx=-\tfrac{1}{2}\pi\mathrm{K}'+\tfrac{1}{2}\mathrm{K}\log\left\{\frac{2k'^{\frac{3}{2}}}{1-k'}\right\}\quad .\ .\quad (58),$$

$$\int_0^{\mathrm{K}}\log\{1+k\,\mathrm{sn}^2 x\}\,dx=-\tfrac{1}{8}\pi\mathrm{K}'+\tfrac{1}{2}\mathrm{K}\log\left\{\frac{2(1+k)}{k^{\frac{1}{2}}}\right\}\quad .\ .\ .\ .\quad (59),$$

$$\int_0^{\mathrm{K}}\log\{1-k\,\mathrm{sn}^2 x\}\,dx=-\tfrac{1}{8}\pi\mathrm{K}'+\tfrac{1}{2}\mathrm{K}\log\left\{\frac{2(1-k)}{k^{\frac{1}{2}}}\right\}\quad .\ .\ .\ .\quad (60),$$

$$\int_0^{\mathrm{K}}\log\{1-k(k-ik')\,\mathrm{sn}^2 x\}\,dx=-\tfrac{1}{8}\pi\mathrm{K}'+\tfrac{1}{2}\mathrm{K}\log\left\{\frac{2k'^{\frac{3}{2}}(k'+ik)}{k^{\frac{1}{2}}}\right\}\quad (61),$$

$$\int_0^{\mathrm{K}}\log\{1-k(k+ik')\,\mathrm{sn}^2 x\}\,dx=-\tfrac{1}{8}\pi\mathrm{K}'+\tfrac{1}{2}\mathrm{K}\log\left\{\frac{2k'^{\frac{3}{2}}(k'-ik)}{k^{\frac{1}{2}}}\right\}\quad (62).$$

In (58) as the integral is so written that its value may be real, the term involving i that enters from (54) has been rejected.

By the addition of (57) and (58) we obtain (17), while (59) and (60) lead in a similar manner to (16), and (61) and (62) to (18). The reason why this happens is easily seen, for since $1-2\,\mathrm{sn}^2 x+k^2\,\mathrm{sn}^4 x$ is the numerator of $\mathrm{cn}\,2x$, we have

$$1-2\,\mathrm{sn}^2 x+k^2\,\mathrm{sn}^4 x=\left\{1-\frac{\mathrm{sn}^2 x}{\mathrm{sn}^2\frac{1}{2}\mathrm{K}}\right\}\left\{1-\frac{\mathrm{sn}^2 x}{\mathrm{sn}^2(\frac{1}{2}\mathrm{K}+i\mathrm{K}')}\right\},$$

which $$=\{1-k^2\,\mathrm{sn}^2 x\,\mathrm{sn}^2(\tfrac{1}{2}\mathrm{K}+i\mathrm{K}')\}\{1-k^2\,\mathrm{sn}^2 x\,\mathrm{sn}^2\tfrac{1}{2}\mathrm{K}\},$$

$$=\{1-(1+k')\,\mathrm{sn}^2 x\}\{1-(1-k')\,\mathrm{sn}^2 x\},$$

and, similarly

$$1-k^2\,\mathrm{sn}^4 x=\{1-k^2\,\mathrm{sn}^2 x\,\mathrm{sn}^2\tfrac{1}{2}i\mathrm{K}'\}\{1-k^2\,\mathrm{sn}^2 x\,\mathrm{sn}^2(\mathrm{K}+\tfrac{1}{2}i\mathrm{K}')\}$$

$$=\{1+k\,\mathrm{sn}^2 x\}\{1-k\,\mathrm{sn}^2 x\},$$

and

$$1-2k^2\,\mathrm{sn}^2 x+k^2\,\mathrm{sn}^4 x$$

$$=\{1-k^2\,\mathrm{sn}^2 x\,\mathrm{sn}^2(\tfrac{1}{2}\mathrm{K}-\tfrac{1}{2}i\mathrm{K}')\}\{1-k^2\,\mathrm{sn}^2 x\,\mathrm{sn}^2(\tfrac{1}{2}\mathrm{K}+\tfrac{1}{2}i\mathrm{K}')\}$$

$$=\{1-k(k-ik')\,\mathrm{sn}^2 x\}\{1-k(k+ik')\,\mathrm{sn}^2 x\}.$$

§ 13. In any integral we may replace $\mathrm{sn}\,x$, $\mathrm{cn}\,x$, $\mathrm{dn}\,x$, by $\sin x$, $\cos x$, $(1-k^2\sin^2 x)^{\frac{1}{2}}$, if we also replace dx by $\dfrac{dx}{(1-k^2\sin^2 x)^{\frac{1}{2}}}$, and make the necessary changes in the limits. This is evident, since

$$\frac{d\,\mathrm{am}\,x}{dx}=\mathrm{dn}\,x,$$

and as $\mathrm{am}\,\mathrm{K}=\frac{1}{2}\pi$, $\mathrm{am}\,2\mathrm{K}=\pi$, we see that to the limits K or 2K and correspond respectively the limits $\frac{1}{2}\pi$ or π and 0.

Thus (5), (6), (7) may be written

$$\int_0^{\frac{1}{2}\pi}\frac{\log\sin x}{(1-k^2\sin^2 x)^{\frac{1}{2}}}\,dx=-\tfrac{1}{4}\pi\mathrm{K}'-\tfrac{1}{2}\mathrm{K}\log k,$$

$$\int_0^{\frac{1}{2}\pi}\frac{\log\cos x}{(1-k^2\sin^2 x)^{\frac{1}{2}}}\,dx=-\tfrac{1}{4}\pi\mathrm{K}'+\tfrac{1}{2}\mathrm{K}\log k,$$

$$\int_0^{\frac{1}{2}\pi}\frac{\log(1-k^2\sin^2 x)}{(1-k^2\sin^2 x)^{\frac{1}{2}}}\,dx=\mathrm{K}\log k',$$

while, for example, (57) becomes

$$\int_0^{\frac{1}{2}\pi}\frac{\log\{1-(1-k')\sin^2 x\}}{(1-k^2\sin^2 x)^{\frac{1}{2}}}\,dx=\tfrac{1}{2}\mathrm{K}\log\left(\frac{2k'^{\frac{1}{2}}}{1+k'}\right),$$

and the other integrals may be similarly transformed.

In this form several of the above integrals have been obtained by Mr. William Roberts in his papers "Sur l'Evaluation de quelques Intégrales Définies par les Fonctions Elliptiques" ("Liouville's Journal," t. xi, 1846, pp. 157—173), "Sur l'Intégrale Définie $\int_0^{\frac{1}{2}\pi}\frac{\log(1+n\sin^2\phi)}{(1-k^2\sin^2\phi)^{\frac{1}{2}}}d\phi$," (Id., pp. 471—476), and "Note sur quelques Intégrales Transcendantes" (Id., t. xii, 1847, pp. 449—456), which contain evaluations equivalent to (5), (6), (7), (16), (38), (44), (48), (57), (59), (60). In the first of these papers the value of the integral in (7) is found by means of the transformation $k'\tan\phi\tan\psi=1$, which is equivalent to the substitution of $\mathrm{K}-x$ for x; and in the last paper the values of the integrals in (5) and (6) are found directly from the q-products, viz.:

$$\mathrm{sn}\left(\frac{2\mathrm{K}x}{\pi}\right)=\frac{2q^{\frac{1}{4}}}{k^{\frac{1}{2}}}\sin x\,\frac{(1-2q^2\cos 2x+q^4)(1-2q^4\cos 2x+q^8)\ldots}{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)\ldots}$$

$$\mathrm{cn}\left(\frac{2\mathrm{K}x}{\pi}\right)=2\left(\frac{k'}{k}\right)^{\frac{1}{2}}q^{\frac{1}{4}}\cos x\,\frac{(1+2q^2\cos 2x+q^4)(1+2q^4\cos 2x+q^8)\ldots}{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)\ldots}$$

by means of the integral (1), which gives

$$\int_0^{\frac{1}{2}\pi}\log(1-2q^n\cos 2x+q^{2n})\,dx=0.$$

The equation (38) is also obtained directly from the q-product for Θ; this is of course practically equivalent to the use of the cosine series (52) for $\log\Theta x$ quoted in § 11.

The last two papers relate chiefly to the integral

$$\int_0^{\mathrm{K}} \log(1+n\,\mathrm{sn}^2 x)\,dx,$$

and one of the methods indicated for obtaining its value depends upon the formula

$$2\Upsilon\,\mathrm{am}\,p + 2\Upsilon\,\mathrm{am}\,q = \Upsilon\,\mathrm{am}\,(p+q) + \Upsilon\,\mathrm{am}\,(p-q) - \log(1-k^2\,\mathrm{sn}^2 p\,\mathrm{sn}^2 q),$$

and is therefore the same in principle as that employed in § 10. Mr. Roberts generally uses the Legendrian notation and the function Υ, for example the formula (44), viz.:

$$\int_0^{\mathrm{K}} \log(1-k^2\,\mathrm{sn}^2 x\,\mathrm{sn}^2 y)\,dx = 2\mathrm{K}\log\frac{\Theta 0}{\Theta y}$$

is written

$$\int_0^{\frac12\pi} \frac{\log(1-k^2\sin^2\theta\sin^2\phi)}{(1-k^2\sin^2\phi)^{\frac12}}\,d\phi = \mathrm{E}(k)[\mathrm{F}(k,\theta)]^2 - 2\mathrm{F}(k)\Upsilon(k,\theta),$$

but he remarks ("Liouville," t. xii, p. 453) that some of the results could have been obtained more readily by the use of Θ instead of Υ.

Mr. Roberts also gives the values of the integrals

$$\int_0^{\frac12\pi} \frac{\log(1+\cot^2\theta\sin^2\phi)}{(1-k^2\sin^2\phi)^{\frac12}}\,d\phi,$$

$$\int_0^{\frac12\pi} \frac{\log\{1-(1-k'^2\sin^2\theta)\sin^2\phi\}}{(1-k^2\sin^2\phi)^{\frac12}}\,d\phi,$$

$$\int_0^{\frac12\pi}\int_0^{\frac12\pi} \frac{\log(1+\cot^2\theta\sin^2\phi)}{(1-k^2\sin^2\phi)^{\frac12}(1-k'^2\sin^2\theta)^{\frac12}}\,d\theta\,d\phi,$$

$$\int_0^{\frac12\pi}\int_0^{\frac12\pi} \frac{\log\{1-(1-k'^2\sin^2\theta)\sin^2\phi\}}{(1-k^2\sin^2\phi)^{\frac12}(1-k'^2\sin^2\theta)^{\frac12}}\,d\theta\,d\phi.$$

These can be derived from (44) by the substitution of $i\mathrm{K}'-iy$ and $\mathrm{K}+i\mathrm{K}'-iy$ in place of y, for

$$k\,\mathrm{sn}(i\mathrm{K}'-iy) = i\,\frac{\mathrm{cn}\,(y,k')}{\mathrm{sn}\,(y,k')},$$

$$k\,\mathrm{sn}\,(\mathrm{K}+i\mathrm{K}'-iy) = \mathrm{dn}\,(y,k'),$$

$$\Theta(i\mathrm{K}'-iy) = \frac{k'^{\frac12}\mathrm{K}^{\frac12}}{\mathrm{K}'^{\frac12}}\,e^{\frac{\pi}{4\mathrm{KK}'}(\mathrm{K}'-y)^2}\,\mathrm{sn}\,(y,k')\Theta(y,k'),$$

$$\Theta(\mathrm{K}+i\mathrm{K}'-iy) = \frac{1}{k^{\frac12}}\,\frac{1}{\mathrm{sn}\,(y,k')}\,\Theta(i\mathrm{K}'-iy),$$

whence we find

$$\int_0^{\mathrm{K}} \log\left\{1+\frac{\mathrm{cn}^2(y,k')}{\mathrm{sn}^2(y,k')}\mathrm{sn}^2 x\right\}dx$$

$$= -\tfrac12\frac{\pi}{\mathrm{K}'}(\mathrm{K}'-y)^2 + \mathrm{K}\log\left(\frac{2\mathrm{K}'}{\pi}\right) - 2\mathrm{K}\log\Theta(y,k') - 2\mathrm{K}\log\mathrm{sn}(y,k') \quad (63),$$

$$\int_0^{\mathrm{K}} \log\{1-\mathrm{dn}^2(y,k')\,\mathrm{sn}^2 x\}dx$$

$$=-\tfrac{1}{2}\frac{\pi}{\mathrm{K}'}(\mathrm{K}'-y)^2+\mathrm{K}\log\left(\frac{2k'\mathrm{K}'}{\pi}\right)-2\mathrm{K}\log\Theta(y,k') \quad . \quad . \quad . \quad (64),$$

and integrating with regard to y between the limits K′ and 0 these give

$$\int_0^{\mathrm{K}}\int_0^{\mathrm{K}'}\log\left\{1+\frac{\mathrm{cn}^2(y,k')}{\mathrm{sn}^2(y,k')}\mathrm{sn}^2 x\right\}dx=\tfrac{1}{3}\pi\mathrm{K}^2-\tfrac{1}{6}\pi\mathrm{K}'^2+\tfrac{1}{3}\mathrm{K}\mathrm{K}'\log\left(\frac{4k'^2}{k}\right) \quad . \quad (65),$$

$$\int_0^{\mathrm{K}}\int_0^{\mathrm{K}'}\log\{1-\mathrm{dn}^2(y,k')\,\mathrm{sn}^2 x\}dx=-\tfrac{1}{6}\pi\mathrm{K}^2-\tfrac{1}{6}\pi\mathrm{K}'^2+\tfrac{1}{3}\mathrm{K}\mathrm{K}'\log\left(\frac{4k'^2}{k}\right) (66),$$

which are in fact Mr. Roberts's results ("Liouville," t. xii, p. 456). In the "Philosophical Magazine," for December, 1860,* Professor Sylvester has given the values of two integrals which are equivalent to (5) and (28). The method depends upon an expansion in a series and is quite different to any employed in this paper. Professor Sylvester notices particularly the fact that the values of the integrals in (5) and (28) should be equal in magnitude.

§ 14. Recently in the "Mathematische Annalen" of Clebsch and Neumann (t. xi, 1877, pp. 567—570), Dr. Enneper has obtained the evaluation

$$\int_0^{\mathrm{K}} \mathrm{dn}^2 x \log \mathrm{dn}\, x dx=\tfrac{1}{2}\mathrm{E}\log k'+\tfrac{1}{2}(1+k'^2)\mathrm{K}-\mathrm{E}$$

by the substitution of $\mathrm{K}-x$ for x and further transformations: but the value of this integral had been previously found in a somewhat similar manner by Mr. Roberts, in a note in "Liouville's Journal" for 1846 ("Extrait d'une Lettre addressée à M. Liouville," t. xi, pp. 343, 344).

Dr. Enneper, in his paper, by the use of (33) shows that

$$\int_0^{\mathrm{K}} \arctan\left\{\frac{k'}{\mathrm{dn}(x+a)\,\mathrm{dn}(x-a)}\right\}dx$$

$$=\int_0^{\mathrm{K}} \arctan\left\{\frac{\mathrm{dn}(x+a)\,\mathrm{dn}(x-a)}{k'}\right\}dx=\tfrac{1}{4}\pi\mathrm{K} \quad . \quad . \quad (67),$$

and it may be observed that, by following the same method, we have at once

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* "Notes to the Meditation on Poncelet's Theorem, including a Valuation of the two new Definite Integrals

$$\int_0^{\frac{\pi}{2}}\frac{\log\cos\phi}{\sqrt{1-b^2(\cos\phi)^2}}d\phi, \qquad \int_0^{\frac{\pi}{2}}\frac{\log\{1+\sqrt{1-b^2(\cos\phi)^2}\}}{\sqrt{1-b^2(\cos\phi)^2}}d\phi,\text{''}$$

"Phil. Mag.," ser. iv, t. xx, pp. 525—533 (1860). See also, by the same author, "Note on certain Definite Integrals," "Quarterly Journal of Mathematics," t. iv, pp. 319—324 (1861).

$$\int_0^{\mathrm{K}} \operatorname{arc\,tan}\left(\frac{\operatorname{dn} x}{k'^{\frac12}}\right)dx=\int_0^{\mathrm{K}} \operatorname{arc\,tan}\left(\frac{k'^{\frac12}}{\operatorname{dn} x}\right)dx$$

$$=\tfrac12\int_0^{\mathrm{K}}\left\{\operatorname{arc\,tan}\left(\frac{\operatorname{dn} x}{k'^{\frac12}}\right)+\operatorname{arc\,tan}\left(\frac{k'^{\frac12}}{\operatorname{dn} x}\right)\right\}dx$$

$$=\tfrac12\int_0^{\mathrm{K}}\tfrac12\pi\,dx=\tfrac14\pi\mathrm{K} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68).$$

Also, n being any quantity, we see from (33) that

$$\int_0^{\mathrm{K}}\frac{dx}{1+\dfrac{\operatorname{dn}^n x}{k'^{\frac12 n}}}=\int_0^{\mathrm{K}}\frac{dx}{1+\dfrac{k'^{\frac12 n}}{\operatorname{dn}^n x}}$$

$$=\tfrac12\int_0^{\mathrm{K}}\left(\frac{k'^{\frac12 n}}{k'^{\frac12 n}+\operatorname{dn}^n x}+\frac{\operatorname{dn}^n x}{k'^{\frac12 n}+\operatorname{dn}^n x}\right)dx$$

$$=\tfrac12\mathrm{K}.$$

so that

$$\int_0^{\mathrm{K}}\frac{dx}{k'^{\frac12 n}+\operatorname{dn}^n x}=\tfrac12\frac{\mathrm{K}}{k'^{\frac12 n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69),$$

$$\int_0^{\mathrm{K}}\frac{\operatorname{dn}^n x}{k'^{\frac12 n}+\operatorname{dn}^n x}dx=\tfrac12\mathrm{K} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70),$$

and, similarly,

$$\int_0^{\mathrm{K}}\frac{k'^n}{k'^n+\operatorname{dn}^n(x-a)\operatorname{dn}^n(x+a)}dx$$

$$=\int_0^{\mathrm{K}}\frac{\operatorname{dn}^n(x-a)\operatorname{dn}^n(x+a)}{k'^n+\operatorname{dn}^n(x-a)\operatorname{dn}^n(x+a)}dx$$

$$=\tfrac12\mathrm{K} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71).$$

Further, if ϕ be an uneven function and n any quantity,

$$\int_0^{\mathrm{K}}\phi\left(\frac{\operatorname{dn} x}{k'^{\frac12}}-\frac{k'^{\frac12}}{\operatorname{dn} x}\right)\log\left(1+\frac{\operatorname{dn}^n x}{k'^{\frac12 n}}\right)dx$$

$$=-\int_0^{\mathrm{K}}\phi\left(\frac{\operatorname{dn} x}{k'^{\frac12}}-\frac{k'^{\frac12}}{\operatorname{dn} x}\right)\log\left(1+\frac{k'^{\frac12 n}}{\operatorname{dn}^n x}\right)dx,$$

whence

$$\int_0^{\mathrm{K}}\phi\left(\frac{\operatorname{dn} x}{k'^{\frac12}}-\frac{k'^{\frac12}}{\operatorname{dn} x}\right)\log\,(k'^{\frac12 n}+\operatorname{dn}^n x)\,dx$$

$$=\tfrac14 n\log k'\int_0^{\mathrm{K}}\phi\left(\frac{\operatorname{dn} x}{k'^{\frac12}}-\frac{k'^{\frac12}}{\operatorname{dn} x}\right)dx+\tfrac12 n\int_0^{\mathrm{K}}\phi\left(\frac{\operatorname{dn} x}{k'^{\frac12}}-\frac{k'^{\frac12}}{\operatorname{dn} x}\right)\log\operatorname{dn} x\,dx,$$

and of course there are other formulæ involving $\dfrac{\operatorname{dn} x}{k'^{\frac12}}$ and $\dfrac{\operatorname{dn}(x-a)\operatorname{dn}(x+a)}{k'}$ which may be proved in a similar manner.

§ 15. It was shown in § 5 that

$$\int_0^{\mathrm{K}}\int_{-\mathrm{K}'}^{\mathrm{K}'} \log \operatorname{cn}(x+iy)\,dx\,dy = \mathrm{KK}' \log \frac{k'}{k};$$

the value of this integral when the limits with regard to y are K′ and 0 may be found as follows.

In the formula

$$\operatorname{cn} u = \frac{k'^{\frac{1}{2}}}{k^{\frac{1}{2}}} q^{\frac{1}{4}} e^{\frac{\pi i u}{2\mathrm{K}}} \frac{\Theta(u+\mathrm{K}+i\mathrm{K}')}{\Theta u},$$

put

$$u = x+iy-\mathrm{K}-i\mathrm{K}',$$

and, taking logarithms, we find

$$\log \operatorname{cn}(x+iy-\mathrm{K}-i\mathrm{K}') = \tfrac{1}{2}\log\frac{k'}{k} - \tfrac{1}{4}\frac{\pi\mathrm{K}'}{\mathrm{K}} + \tfrac{1}{2}\frac{\pi i}{\mathrm{K}}(x+iy-\mathrm{K}-i\mathrm{K}')$$

$$+\log\Theta(x+iy) - \log\Theta(x+iy-\mathrm{K}-i\mathrm{K}') \quad . \quad . \quad . \quad (72)$$

Now $$\int_0^{\mathrm{K}}\int_0^{\mathrm{K}'} \phi(x+iy-\mathrm{K}-i\mathrm{K}')\,dx\,dy = \int_0^{\mathrm{K}}\int_0^{\mathrm{K}'} \phi(x+iy)\,dx\,dy$$

if ϕ be an even function, so that (72) gives

$$\int_0^{\mathrm{K}}\int_0^{\mathrm{K}'} \log \operatorname{cn}(x+iy)\,dx\,dy = \tfrac{1}{2}\mathrm{KK}'\log\frac{k'}{k} - \tfrac{1}{4}\pi\mathrm{K}'^2$$

$$+\tfrac{1}{2}\frac{\pi i}{\mathrm{K}}(\tfrac{1}{2}\mathrm{K}^2\mathrm{K}' + \tfrac{1}{2}i\mathrm{KK}'^2 - \mathrm{K}^2\mathrm{K}' - i\mathrm{KK}'^2)$$

$$= \tfrac{1}{2}\mathrm{KK}'\log\frac{k'}{k} - \tfrac{1}{4}i\pi\mathrm{KK}' \quad . \quad . \quad . \quad . \quad . \quad (73).$$

§ 16. I conclude with the determination of the values of the integrals

$$\int_0^{\infty} e^{-x^2}\Theta\left\{2x\left(\frac{n\mathrm{KK}'}{\pi}\right)^{\frac{1}{2}}\right\}dx, \qquad \int_0^{\infty} xe^{-x^2}\mathrm{H}\left\{2x\left(\frac{n\mathrm{KK}'}{\pi}\right)^{\frac{1}{2}}\right\}dx$$

when n is a positive integer.

$$\Theta\left(\frac{2\mathrm{K}x}{\pi}\right) = 1 - 2q\cos 2x + 2q^4\cos 4x - 2q^9\cos 6x + \text{ \&c.},$$

whence

$$e^{-x^2}\Theta\left(\frac{2\mathrm{K}ax}{\pi}\right) = e^{-x^2} - 2qe^{-x^2}\cos 2ax + 2q^4e^{-x^2}\cos 4ax - \text{ \&c.}$$

Now $$\int_0^{\infty} e^{-x^2}\cos 2bx\,dx = \tfrac{1}{2}\pi^{\frac{1}{2}}e^{-b^2},$$

so that we have

$$\int_0^{\infty} e^{-x^2}\Theta\left(\frac{2\mathrm{K}ax}{\pi}\right)dx = \tfrac{1}{2}\pi^{\frac{1}{2}}(1 - 2qe^{-a^2} + 2q^4e^{-4a^2} - 2q^9e^{-9a^2} + \text{ \&c.})$$

Let $$e^{-a^2}=q^n,$$
n being a positive integer, whence
$$a=\left(\frac{n\pi K'}{K}\right)^{\frac{1}{2}},$$
and
$$\int_0^\infty e^{-x^2}\Theta\left\{2x\left(\frac{nKK'}{\pi}\right)^{\frac{1}{2}}\right\}dx=\tfrac{1}{2}\pi^{\frac{1}{2}}(1-2q^{n+1}-2q^{4(n+1)}+2q^{9(n+1)}-\ \&c.)$$
$$=\tfrac{1}{2}\pi^{\frac{1}{2}}\left(\frac{2k'_{n+1}K_{n+1}}{\pi}\right)^{\frac{1}{2}}$$
$$=\left(\frac{k'_{n+1}K_{n+1}}{2}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74),$$
where k'_{n+1}, K_{n+1} are what k and K become when q is replaced by q^{n+1}, so that k_{n+1} is the modulus obtained by the first real transformation of the $(n+1)^{\text{th}}$ order.

As a particular case put $n=1$, and the formula (74) gives
$$\int_0^\infty e^{-x^2}\Theta\left\{2x\left(\frac{KK'}{\pi}\right)^{\frac{1}{2}}\right\}dx=(\tfrac{1}{2}Kk'^{\frac{1}{2}})^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (75).$$

To evaluate the other integral, multiply the equation
$$H\left(\frac{2Kax}{\pi}\right)=2q^{\frac{1}{4}}\sin ax-2q^{\frac{9}{4}}\sin 3ax+2q^{\frac{25}{4}}\sin 5ax-\ \&c.$$
by e^{-x^2} and integrate, replacing the integrals on the right hand side by their values from the formula
$$\int_0^\infty xe^{-x^2}\sin 2bx\,dx=\tfrac{1}{2}\pi^{\frac{1}{2}}be^{-b^2},$$
we thus find
$$\int_0^\infty xe^{-x^2}H\left(\frac{2Kax}{\pi}\right)dx=\tfrac{1}{2}\pi^{\frac{1}{2}}a(q^{\frac{1}{4}}e^{-\frac{1}{4}a^2}-3q^{\frac{9}{4}}e^{-\frac{9}{4}a^2}+5q^{\frac{25}{4}}e^{-\frac{25}{4}a^2}-\ \&c.)$$
Put, as before, $$e^{-a^2}=q^n,$$
n being a positive integer, then
$$a=\left(\frac{n\pi K'}{K}\right)^{\frac{1}{2}},$$
and the equation becomes
$$\int_0^\infty xe^{-x^2}H\left\{2x\left(\frac{nKK'}{\pi}\right)^{\frac{1}{2}}\right\}dx=\tfrac{1}{2}\pi\left(\frac{nK'}{K}\right)^{\frac{1}{2}}(q^{\frac{1}{4}(n+1)}-3q^{\frac{9}{4}(n+1)}+5q^{\frac{25}{4}(n+1)}-\ \&c.)$$

But $$2q^{\frac{1}{4}}-6q^{\frac{9}{4}}+10q^{\frac{25}{4}}-\ \&c.=\left\{kk'\left(\frac{2K}{\pi}\right)^3\right\}$$

(" Fundamenta Nova," p. 184), so that

$$\int_0^\infty xe^{-x^2}\mathrm{H}\left\{2x\left(\frac{n\mathrm{KK}'}{\pi}\right)^{\frac{1}{2}}\right\}dx=\left\{\tfrac{1}{2}\frac{n\mathrm{K}'}{\pi\mathrm{K}}k_{n+1}k'_{n+1}\mathrm{K}^3{}_{n+1}\right\}^{\frac{1}{2}} \quad . \quad (76).$$

Put $n=1$ and this formula gives

$$\int_0^\infty xe^{-x^2}\mathrm{H}\left\{2x\left(\frac{\mathrm{KK}'}{\pi}\right)^{\frac{1}{2}}\right\}dx=\tfrac{1}{2}\mathrm{K}k\left(\tfrac{1}{2}\frac{\mathrm{K}'k'^{\frac{1}{2}}}{\pi}\right)^{\frac{1}{2}} \quad . \quad . \quad (77).$$

IX. " Values of the Theta and Zeta Functions for certain Values of the Argument." By J. W. L. GLAISHER, M.A., F.R.S., Fellow of Trinity College, Cambridge. Received July 31, 1879.

§ 1. In § 12 of the preceding paper " On Definite Integrals involving Elliptic Functions," it was necessary to determine the values of $\Theta(\frac{1}{2}\mathrm{K})$, $\Theta(\frac{1}{2}\mathrm{K}+i\mathrm{K}')$, $\Theta(\frac{1}{2}i\mathrm{K}')$, $\Theta(\mathrm{K}+\frac{1}{2}i\mathrm{K}')$, and $\Theta^4(\frac{1}{2}\mathrm{K}+\frac{1}{2}i\mathrm{K}')$, which were required in the evaluation of some of the integrals. This led me to calculate a table of the values of the Θ and H functions when the arguments were of the form $\mathrm{K}+ni\mathrm{K}'$, for the values 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, of m and n, and the results are contained in this paper. For the sake of completeness the corresponding values of the Z function are also given; and some remarks connected with the q-series to which the formulæ lead are added.

A table of the values of the sn, cn, dn for the above-mentioned arguments is given by Professor Cayley on page 74 of his " Elementary Treatise on Elliptic Functions " (1876) : this table* is so useful that it seemed desirable to supplement it by a similar one for the Θ, H, and Z functions.

§ 2. The values found in § 13 are

$$\Theta(\tfrac{1}{2}\mathrm{K}) \qquad =\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k'^{\frac{1}{8}}(1+k')^{\frac{1}{4}},$$

$$\Theta(\tfrac{1}{2}\mathrm{K}+i\mathrm{K}') \quad =q^{-\frac{1}{4}}\frac{\mathrm{K}^{\frac{1}{2}}}{2^{\frac{1}{4}}\pi^{\frac{1}{2}}}k'^{\frac{1}{8}}(1-k')^{\frac{1}{4}}(1+i),$$

$$\Theta(\tfrac{1}{2}i\mathrm{K}') \qquad =q^{-\frac{1}{16}}\frac{2^{\frac{1}{4}}\mathrm{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{8}}(1-k)^{\frac{1}{4}},$$

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* I may here note that the value of cn $(\frac{1}{2}\mathrm{K}+\frac{1}{2}i\mathrm{K}')$ should be $\frac{1-i}{\sqrt{2}}\frac{\sqrt{k'}}{\sqrt{k}}$ instead of $-\frac{1-i}{\sqrt{2}}\frac{\sqrt{k'}}{\sqrt{k}}$.